SEPARABLE L_1 PREDUALS ARE QUOTIENTS OF $C(\Delta)$

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ABSTRACT

It is proved that every separable predual space of an L_1 space is a quotient space of $C(\Delta)$.

1. Introduction

Recently Benyamini and Lindenstrauss [1] gave an example of a separable L_1 predual which is not isomorphic to a complemented subspace of any $C(K)$ space. Thus the result proved here—that is, if X is a separable Banach space with X^* isometric to $L_1(\mu)$ for some measure μ, then X is isometric to a quotient space of the continuous real-valued functions on the Cantor set—this result is the best possible for relating general separable L_1 preduals to $C(K)$ spaces. However, the more interesting problem of whether every separable $\mathscr{L}_\infty$ space is isomorphic to a quotient of a $C(K)$ space, remains open (refer to [3]).

One immediate application of our theorem and results of Pełczyński [5] and Rosenthal [6] we can state in a corollary.

COROLLARY. *Suppose X is an L_1 predual and T is a non-weakly compact operator from X into some Banach space Y.*

(i) *There is a subspace W of X isomorphic to c_0 and such that the restriction of T to W is an isomorphism.*

(ii) *If X is separable but T^*Y^* is non-separable, then there is a subspace Z of X isomorphic to $C(\Delta)$ such that the restriction of T to Z is an isomorphism.*

We use standard Banach space theory notation. As is customary in the study of

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L_1 preduals, we consider real Banach spaces only so as to avoid purely technical complications that arise in the complex case.

2. The construction

It is known that if X is an L_1 predual then X is the completion of the union of subspaces $E_1 \subseteq E_2 \subseteq \cdots$ with E_n isometric to l^n_∞. (Refer to [2] and [4].) Further, bases $(e_i^n)_{i=1}^n$ for E_n may be chosen so that, for each n and i,

(1) $$e_i^n = e_i^{n+1} + \beta_i^n e_{n+1}^{n+1},$$

(2) $$\sum_{j=1}^{n} |\beta_j^n| \leqq 1, \text{ and}$$

(3) $$\left\| \sum_{j=1}^{n} \alpha_j e_j^n \right\| = \max |\alpha_j| \text{ for any scalars } (\alpha_j).$$

Now pick a set $\{A_i^n: 1 \leqq i \leqq k_n; n = 1,2,\cdots\}$ of non-empty open and closed subsets of the Cantor set Δ to satisfy

(4) $$k_1 = 1;\ k_{n+1} = 2k_n + 1 \text{ for } n > 1,$$

(5) $$A_1^1 = \Delta,$$

(6) $$A_i^n = A_{2i-1}^{n+1} \cup A_{2i}^{n+1} \text{ for } 1 \leqq i < k_n,$$

(7) $$A_{k_n}^n = A_{2k_n-1}^{n+1} \cup A_{2k_n}^{n+1} \cup A_{k_{n+1}}^{n+1},$$

(8) $\{A_i^n: 1 \leqq i \leqq k_n\}$ is pairwise disjoint,

(9) $\{A_i^n: 1 \leqq i \leqq k_n, n = 1,2,\cdots\}$ is a base for the topology on Δ.

We identify a subset of Δ with its characteristic function. Setting $F_n = \mathrm{Span}\{A_i^n: 1 \leqq i \leqq k_n\}$, we have from (9) that $\bigcup F_n$ is dense in $C(\Delta)$. Now the theorem will follow if we can define a linear operator $T: \bigcup F_n \to \bigcup E_n$ so that T maps the unit ball of $\bigcup F_n$ onto the unit ball of $\bigcup E_n$, for then the continuous extension of T to $C(\Delta)$ will be a quotient mapping of $C(\Delta)$ onto X.

We define T by induction. Let $T\alpha A_1^1 = \alpha e_1^1$ for each scalar α. Suppose that T has been defined on F_n so that $T(\mathrm{Ball}\, F_n) = \mathrm{Ball}(E_n)$. Write $T A_i^n = \sum_{j=1}^n \alpha_j^i e_j^n$ for $1 \leqq i \leqq k_n$. Now $\| \sum_{i=1}^{k_n} \gamma_i A_i^n \| = 1$ for all choices of signs $\gamma_i = \pm 1$, and $\|T\| = 1$, so from (3) it follows that

(10) $$\sum_{i=1}^{k_n} |\alpha_j^i| \leqq 1 \text{ for each } 1 \leqq j \leqq n.$$

Extend T to F_{n+1} by defining

$$(11) \qquad T A^{n+1}_{2i-1} = \sum_{j=1}^{n} \alpha^i_j e^{n+1}_j \text{ for } 1 \leqq i \leqq k_n,$$

$$(12) \qquad T A^{n+1}_{2i} = \sum_{j=1}^{n} \beta^n_j \alpha^i_j e^{n+1}_{n+1} \text{ for } 1 \leqq i < k_n,$$

$$(13) \qquad \begin{aligned} T A^{n+1}_{2k_n} &= 2^{-1}\left(\sum_{j=1}^{n} \beta^n_j \alpha^{k_n}_j - \left[1 - \sum_{i=1}^{k_n-1}\left|\sum_{j=1}^{n} \beta^n_j \alpha^i_j\right|\right]\right) e^{n+1}_{n+1}, \\ T A^{n+1}_{k_{n+1}} &= 2^{-1}\left(\sum_{j=1}^{n} \beta^n_j \alpha^{k_n}_j + \left[1 - \sum_{i=1}^{k_n-1}\left|\sum_{j=1}^{n} \beta^n_j \alpha^i_j\right|\right]\right) e^{n+1}_{n+1}, \end{aligned}$$

and extending T linearly to F_{n+1}. (Of course, to make sure that this new definition of T agrees with the old definition on F_n, one must verify that

$$T A^{n+1}_{2i-1} + T A^{n+1}_{2i} = \sum_{j=1}^{n} \alpha^i_j e^n_j (1 \leqq i < k_n)$$

and

$$T A^{n+1}_{2k_n-1} + T A^{n+1}_{2k_n} + T A^{n+1}_{k_{n+1}} = \sum_{j=1}^{n} \alpha^{k_n}_j e^n_j.$$

We omit this trivial computation.)

Suppose A is an extreme point of the unit ball of F_{n+1}, say $A = \sum_{i=1}^{k_{n+1}} \gamma_i A^{n+1}_i$ with $\gamma_i = \pm 1$ for $1 \leqq i \leqq k_{n+1}$. Write $T A = \sum_{j=1}^{n+1} \Delta_j e^{n+1}_j$. Then from (10) and (11) it follows that for $1 \leqq j \leqq n$, $|\Delta_j| \leqq \sum_{i=1}^{k_n} |\alpha^i_j| \leqq 1$. Now if $\gamma_{k_{n+1}} = \gamma_{2k_n}$ we have from (12) and (13) the estimate

$$\begin{aligned} |\Delta_{n+1}| &\leqq \sum_{i=1}^{k_n-1}\left|\sum_{j=1}^{n} \beta^n_j \alpha^i_j\right| + \left|\sum_{j=1}^{n} \beta^n_j \alpha^{k_n}_j\right| \\ &\leqq \sum_{j=1}^{n} |\beta^n_j| \sum_{i=1}^{k_n} |\alpha^i_j| \\ &\leqq \sum_{j=1}^{n} |\beta^n_j| \text{ by (10)} \\ &\leqq 1 \text{ by (2).} \end{aligned}$$

If $\gamma_{k_{n+1}} = -\gamma_{2k_n}$ we have from (12) and (13) that

$$|\Delta_{n+1}| \leqq \sum_{i=1}^{k_n-1}\left|\sum_{j=1}^{n} \beta^n_j \alpha^i_j\right| + 1 - \sum_{i=1}^{k_n-1}\left|\sum_{j=1}^{n} \beta^n_j \alpha^i_j\right| = 1.$$

Thus by (3), $\| T \| \leqq 1$ on F_{n+1}.

Finally, suppose x is in the unit ball of E_{n+1},

$$\text{say } x = \sum_{i=1}^{n+1} \alpha_i e_i^{n+1} \text{ with } |\alpha_i| \leqq 1.$$

By our inductive hypothesis there is $B \in \text{Ball } F_n$ satisfying $T B = \sum_{i=1}^{n} \alpha_i e$, say $B = \sum_{i=1}^{k_n} \gamma_{2i-1} A_i^n$. Thus by (11) $T \sum_{i=1}^{k_n} \gamma_{2i-1} A_{2i-1}^{n+1} = \sum_{i=1}^{n} \alpha_i e_i^{n+1}$. Now set

$$\gamma_{2i} = \alpha_{n+1} \operatorname{sgn} \left(\sum_{j=1}^{n} \beta_j^n \alpha_j^i \right) \quad (1 \leqq i < k_n)$$

$$\gamma_{2k_n} = -\alpha_{n+1}, \ \gamma_{k_{n+1}} = \alpha_{n+1}.$$

Then $|\gamma_i| \leqq 1$ for each i and

$$T \sum_{i=1}^{k_{n+1}} \gamma_i A_i^{n+1} = x.$$

Thus T Ball F_{n+1} = Ball E_{n+1}. This completes the inductive construction of T and the proof of the theorem.

3. Proof of the corollary

(i) Since the restriction of T to some separable subspace of X is not weakly compact and since every separable subspace of X is contained in a larger subspace which is itself an L_1 predual, we may assume that X is separable. Let $Q: C(\Delta) \to X$ be a quotient map. Then Q^* is an isometry, hence $Q^*T^* = (TQ)^*$ is not weakly compact, whence TQ is not weakly compact. By a result of Pełczyński [5] there is a subspace Y of $C(\Delta)$ with Y isomorphic to c_0 and $TQ_{|Y}$ an isomorphism, thus $W = QY$ has the desired property.

(ii) follows in a similar fashion from Rosenthal's theorem [6] that if S is a operator from $C(\Delta)$ such that S^* has non-separable range, then there is a subspace Y of $C(\Delta)$ such that $S_{|Y}$ is an isomorphism.

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